Self-Similar Law of Energy Release before Materials Fracture

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Abstract

A general law of energy release is derived for stressed heterogeneous materials, being valid from the starting moment of loading till the moment of materials fracture. This law is obtained by employing the extrapolation technique of the self-similar approximation theory. Experiments are accomplished measuring the energy release for industrial composite samples. The derived analytical law is confronted with these experimental data as well as with the known experimental data for other materials.

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1 Introduction

The study of global failure of stressed heterogeneous materials is of great importance for the possibility to control the state of health of complex structures in industry. The majority of industrial materials are composites that, despite their regular appearance at a macroscopic level, intrinsically possess numerous mesoscopic defects, i.e. defects that are neither microscopic nor macroscopic, but which are intermediate between the latter, say at the scale of fibres. These intrinsic imperfections give rise to high local stress concentrations when such composite materials are strongly loaded. Contrary to perfect crystals, whose failure occurs suddenly, due to the cleavage effect with unstable crack propagation, the failure of nonperfect materials develops as a multistep procedure through a succession of local events corresponding to diffuse damage. The progressive growth of the local damages, such as crack nucleation, fibre-matrix debonding and rupture, and matrix separations, culminate in the catastrophic rise of a global crack implying the materials fracture. The gradual process of the defect accumulation, developing before the materials failure, is accompanied by acoustic emission characterizing the energy released from the stressed sample (Broberg, 1999).

To understand the general law of the energy release from composite materials would be of high importance for many applications and could be employed for predicting the fracture of loaded industrial structures. Unfortunately, the general form of such a law for the energy release, describing the behaviour of the latter from the very beginning of loading till the final moment of fracture, is not known. It is the aim of the present paper to derive such a general law of energy release and to confront it with experimental data.

2 Physical picture of rupture

We consider the failure of stressed composite materials, which occurs through the gradual accumulation of local mesoscopic defects, accompanied by a measurable energy release. The physical picture of the process, developing in the *close vicinity* of materials fracture is more or less generally accepted. The fracturing of stressed composite materials is somewhat analogous to a critical phenomenon happening at a second-order phase transition (Herrmann and Roux, 1990). The moment of rupture is similar to a critical point, so that the fracturing process can be described by a renormalization-group scheme (Anifrani *et al.*, 1995). In the vicinity of the critical point of rupture, there exists a critical region (Lamaghére *et al.*, 1996; Sornette and Andersen, 1998), where the energy release can be characterized by a power law decorated by log-periodic oscillations (Sornette, 1998). Such oscillations are related to complex exponents that appear in renormalization-group solutions for critical phenomena (Nauenberg, 1975).

The multistep character of arising mesoscopic defects, reflecting the discrete nature of the fracturing process, makes it possible to compare the diffusion of damages with the diffusion on random lattices (Bernasconi and Schneider, 1983; Dietrich and Sornette, 1998). This analogy has been the starting idea for extensive numerical simulations

applied to the lattice models of fracture (Sahimi and Arbabi, 1996), which predicted the existence of scaling laws in the vicinity of fracture. With regard to real heterogeneous materials, it is important to emphasize two principal points:

First, we keep in mind the composite materials with a random spatial distribution of defects. Because of this spatial randomness, the growth of damages under a uniform loading happens randomly across the volume of the sample. A sufficiently random distribution of defects is naturally related to a high degree of their concentration. The stronger is the degree of imperfection, the more accurate are the scaling laws in the vicinity of rupture and the closer is the similarity of the latter with a critical phenomenon (Johansen and Sornette, 2000).

Second, not only the growth of numerous damages happens randomly in space, but the local growth of each of them randomly stops due to the internal structure of matter. For instance, in a random porous material, the growth of a crack ends as soon as it encounters a pore. In the case of composite materials, there are three main types of damages: matrix cracking, fibre rupture, and matrix-fibre debonding. These damages grow simultaneously and, being intermixed, they create a spatially random distribution of defects. It is worth recalling that even in brittle composite materials, where the fracture occurs rather quickly, the dynamic stress intensity factor at the crack tip dramatically reduces as soon as the crack encounters a fibre, which leads to the crack arrest (Broberg, 1999).

In this way, in a stressed composite material, with the described properties, there arises the proliferation of randomly distributed defects. Close to the point of rupture, the concentration of defects is so high that they form a random self-organized structure. The vicinity of rupture is analogous to the critical region, and the global materials failure corresponds to a critical point. Then, being based on renormalization-group arguments, one comes to the conclusion that the energy release close to the point of rupture follows a power law decorated by log-periodic oscillations (Sornette, 1998).

Although the asymptotic behaviour of the energy release in the close vicinity of materials rupture can be written explicitly, its general law in the whole region, starting from the absence of any load till the moment of rupture, is not known. But, as is clear, the knowledge of this general law would be of great importance. For instance, knowing such a law, one could, on the basis of early observations, predict the following behaviour of a stressed sample and even forecast the moment of its fracture.

In the recent communication (Moura and Yukalov, 2002), we suggested a procedure for deriving the general law of energy release by employing the self-similar approximation theory (Yukalov *et al.*, 1990, 1991, 1992, 1997-a-b, 1998-a-b). However, for this purpose, we considered only the simplest one-step variant of the procedure, which, though allowing for the extrapolation outside the critical region, could not provide a very accurate description far from the latter region, especially at the very beginning of the loading process. In the present paper, we invoke a more elaborate variant of the self-similar approximation theory for deriving a really general law of energy release for stressed composite materials, the law that would be valid starting from zero load till the moment of fracture.

Before passing to technical details it is important to clarify several points:

First of all, we consider here the energy release as the main characteristic describing the process of fracturing. This is because the energy release is the quantity that can be *directly measured* in acoustic emission experiments. In principle, one could invoke some other characteristics describing the evolution of the material structure. Thus, in the classical models of continuum damage mechanics, one considers a quantity playing the role of a material-structure parameter (Kachanov, 1986). The structure parameter can be rather convenient for describing the spatial state of continuous medium. It could also be useful at the very beginning of the fracturing process in heterogeneous non-plastic materials. However, the continuous mechanism is no longer valid when the studied material becomes effectively discontinuous at the advanced stage of fracturing, owing to high concentration of random defects. Even if it would be possible to attribute a kind of structural order parameter to composite materials, as those we consider, we prefer to deal with the energy release. This is not because other characteristics are not admissible, but just because the energy release is a directly measurable quantity, hence, it is the most convenient for comparison with experiments.

The oscillatory behaviour of the energy release in the vicinity of fracture can be presented in several slightly different ways. We opt here for the so-called log-periodic form suggested and studied in several papers (Anifrani *et al.*, 1995; Johansen and Sornette, 2000). This log-periodic form can be justified by explicit mechanical models (Dietrich and Sornette, 1998; Sornette and Andersen, 1998). The models of rupture yielding the log-periodicity, as well as the physical interpretation of the related parameters, have been analysed in the recent review by Sornette (1998), because of which there is no reason to repeat them here.

One should not forget that the log-periodic presentation models the behaviour of the energy release only in the vicinity of rupture, thus, being invalid far from the fracture point. In order to extrapolate this log-periodic form to the whole process of loading, starting from the very beginning till the moment of fracture, one needs to employ some extrapolation techniques. For this purpose, we use the *self-similar approximation theory* (Yukalov, 1990, 1991, 1992; Yukalov *at al.*, 1997-a-b, 1998-a-b). To make this paper more self-consistent, we describe the main ideas of this theory in Appendix.

It is worth emphasizing that the employed self-similar extrapolation technique is quite general and can be used for extrapolating a variety of functions, but not solely those having the sinusoidal log-periodic form. For instance, there exists a wide class of parametric homogeneous functions, including as a particular case the sinusoidal log-periodic functions (Borodich, 1997, 1998-a-b; Borodich and Galanov, 2002). If the behaviour of the energy release in the close vicinity of fracture could be presented through such a parametric homogeneous function, one could also extrapolate this behaviour by employing the self-similar approximation theory. Our paper provides a general method of such an extrapolation. The usage of the sinusoidal log-periodic presentation is not compulsory, but just it is motivated by several known models of fracture (Sornette, 1998).

3 Derivation of general law

The technique of the derivation is as follows. We start from the critical region of rupture, where the energy release is known to be asymptotically presented by a power law decorated by log-periodic oscillations (Sornette, 1998). Then, we extrapolate this asymptotic form outside the critical region by means of the self-similar approximation theory (Yukalov *et al.*, 1990, 1991, 1992, 1997-a-b, 1998-a-b). This theory provides a general tool for extrapolating asymptotic expansions to the whole region of variables, and it has been successfully applied to describing critical phenomena as well as to time-series analysis (Yukalov *et al.*, 1999, 2000, 2001). All technical details of the self-similar approximation theory can be found in the cited references. For the convenience of the reader, the basic ideas of this approach are sketched in Appendix.

To correctly formulate the notion of the group self-similarity on the manifold of approximations, we have, first of all, to present the sought physical quantity in a scale-invariant form. To this end, we need, first, to choose a convenient dimensionless variable for the energy release. The latter can be considered as a function of the uniformly and isotropically imposed pressure p changing from $p = 0$ to the critical pressure of rupture p_c. In the case of an anisotropic load, one can measure the energy release as a function of elongation l, with l varying from zero to a critical elongation l_c, when the stressed material breaks. One may also treat the energy release as a function of time varying from the initial moment $t = 0$, when there is no load, till the critical moment t_c, when the loaded sample is fractured. In each of such cases, we can introduce a relative dimensionless variable defined by one of the following forms:

$$x \equiv \left\{ \frac{p_c - p}{p_c} \, , \frac{l_c - l}{l_c} \, , \frac{t_c - t}{t_c} \right\} \, , \tag{1}$$

with x given in the domain $[0, 1]$. Then, if $E(x)$ is a cumulative energy release, changing from $E(1) = 0$ at the start of the loading process to $E(0) = E_c$ at the moment of rupture, we define the reduced energy release

$$f(x) \equiv \frac{E(x)}{E(0)} \, . \tag{2}$$

By this definition, function (2) satisfies the boundary conditions

$$f(0) = 1 \, , \qquad f(1) = 0 \, . \tag{3}$$

There are two other physical conditions

$$f(x) \geq 0 \, , \qquad f'(x) \leq 0 \qquad (0 \leq x \leq 1) \, , \tag{4}$$

requiring that both the cumulative energy release and the energy release rate be positive. Sometimes, it is useful to invoke the integral normalization condition

$$\int_0^1 f(x) \, dx = \Phi \, , \tag{5}$$

in which the value Φ is prescribed by the considered experiment.

In the asymptotic vicinity of the critical point $x = 0$, the dimensionless energy release (2) follows, as is discussed in the previous section, the power law decorated by log-periodic oscillations, which reads

$$f(x) \simeq 1 + a_1\, x^{\alpha} + a_2\, x^{\alpha} \cos(\omega \ln x + \varphi) \; , \tag{6}$$

as $x \to 0$, where $a_1,\ a_2,\ \omega,\ \alpha$, and φ are the parameters characterizing the studied material, because of which they can be called the *material parameters*. The most physically important here is the parameter ω characterizing the frequency of the log-periodic oscillations, while a_2 describes their amplitude. As is explained above, form (6) follows from some models of fracture (Sornette, 1998).

Function (6) can be treated as the real part of the complex expression

$$F(x) \simeq 1 + A_1\, x^{\alpha} + A_2\, x^{\alpha + i\omega} \; , \tag{7}$$

in which $x \to 0$ and the coefficients are

$$A_1 \equiv a_1 \; , \qquad A_2 \equiv a_2\, e^{i\varphi} \; .$$

Expansion (7) can be renormalized employing the self-similar approximation theory (Yukalov *et al.*, 1990, 1991, 1992, 1997-a-b, 1998-a-b). For this purpose, each term in Eq. (7) is considered as a correction to the sum of the previous terms. The self-similar renormalization of Eq. (7), based on the self-similar exponential approximants (Yukalov and Gluzman, 1998-a), results in

$$F^*(x) = \exp\left(c_1 x^{\alpha}\ \exp\left(c_2 x^{i\omega} \right) \right) \; , \tag{8}$$

where

$$c_1 \equiv A_1 \tau_1 \; , \qquad c_2 \equiv \frac{A_2}{A_1}\, \tau_1$$

are the so-called controllers, with τ_1 and τ_2 being the step-control functions to be defined from optimization conditions. Since the sought function $f(x)$ is the real part of Eq. (7), then the real part of Eq. (8) provides us with the self-similar approximant

$$f^*(x) = \mathrm{Re}\; F^*(x) \; . \tag{9}$$

Taking the real part of Eq. (8), it is convenient to pass to the notation

$$c_1 \equiv c \; , \qquad c_2 \equiv a + i\, b \; ,$$

in which all parameters $a,\ b$, and c are real. In this way, we obtain

$$f^*(x) = \cos\left[c x^{\alpha} \sin g(x)\ \exp h(x) \right]\ \exp\left[c x^{\alpha} \cos g(x)\ \exp h(x) \right] \; , \tag{10}$$

where we introduce the functions

$$g(x) \equiv a \sin(\omega\ \ln x) + b \cos(\omega\ \ln x) \; , \qquad h(x) \equiv a \cos(\omega\ \ln x) - b \sin(\omega\ \ln x) \; .$$

Expression (10) is the sought general law for the energy release, extrapolating the asymptotic expansion (6) to the whole region of the variable $x \in [0, 1]$. Because $f^*(0) = 1$, the first of the boundary conditions (3) is automatically valid. The second of these boundary conditions requires the validity of the equation

$$c\, e^a \sin b = \frac{\pi}{2} + \pi n \; ,$$

in which n is an integer. Since the latter is arbitrary, this condition does not impose a strict constraint, so that in what follows five material parameters $a,\ b,\ c,\ \alpha$, and ω can be treated as independent. The values of these parameters can be found by comparing the law (10) with experiments. The physical meaning of these parameters is clear from expression (10): the parameter ω describes the log-periodic frequency of oscillations; the parameters a and b, their amplitude; and the parameters c and α characterize the rate of the overall increase of the released energy.

4 Comparison with experimental data

We accomplished experiments studying the behaviour of the cumulative energy release as a function of the dimensionless parameter (1). The studied samples were the glass-polyester composite plates of the sizes $2 \times 14 \times 120$ mm. Each plate contained 75% of unidirectional fibres of diameter 20 μm. At the initial moment, a plate was subject to a steady tensile stress of 650 N at an angle of 27^o to the fibre direction, which forced the material to creep. Varying the angle does not result in the principal change of the process. During the creeping test, both acoustic emission and elongation were recorded up to the global fracture. Piezoelectric transducers recording the Lamb waves, i.e. guided waves in the plate, were employed. Electric signals, reflecting the energy release rate caused by sudden local damages, were used for calculating the cumulative energy release. The stressed sample was inside an oven supporting a constant temperature of 60^oC. The general experimental setup is shown in Fig. 1 and more details are described by Moura *et al.* (2002). Particular characteristics of these experiments could be varied, which, however, would not essentially influence the final results of measurements, provided these results are presented in the scale-invariant dimensionless form (2). A typical behaviour of the dimensionless cumulative energy release (2) versus the dimensionless elongation (1), observed in our experiments, is shown in Fig. 2, where it is compared with the law (10); the fitting material parameters being $a = 0.084$, $b = -0.364$, $c = -4.05$, $\alpha = 0.613$, and $\omega = 1.49$.

Note that the derivative of $f^*(x)$, corresponding to the energy release rate, diverges at the point of global failure, which confirms the analogy of the latter with a critical point. We may also define the critical index

$$\Delta \equiv \lim_{x \to +0} \frac{\ln |f'(x)|}{\ln x} \; ,$$

for which we find $\Delta = \alpha - 1 = -0.387$.

Among other known experiments on the energy release measurements, the most reliable are those accomplished by Anifrani *et al.* (1995) who recorded the acoustic emission during the uniform pressurization of spherical tanks of kevlar and carbon fibres impregnated with resin. This composite material was wrapped up around a thin metallic liner (steal or titanium). The samples were fabricated by Aérospatial-Matra Inc.. The general experimental methodology was similar to ours, with the difference that the dimensionless variable (1) represented the change of pressure. In Fig. 3, we compare the law (10) with typical acoustic-emission measurements accomplished by Anifrani *et al.* (1995) and reported on the left top of figure 3 in the paper by Johansen and Sornette (2000). The material fitting parameters are $a = -0.101$, $b = -0.131$, $c = -10.86$, $\alpha = 0.839$, and $\omega = 3.16$. Again, the rate $f'(x)$ diverges at the critical point of fracture, with the critical index $\Delta = -0.161$.

5 Conclusion

A general law of energy release for stressed composite materials is derived, being valid in the whole interval starting from the initially imposed load till the very moment of fracture. This law is presented by the scale-invariant dimensionless form (10). The derivation of formula (10) is based on the extrapolation technique of the self-similar approximation theory. In the present case, we extrapolate the log-periodically decorated power law that is asymptotically valid only in the vicinity of materials rupture. A specific feature of the present extrapolation is the occurrence of complex-valued exponents.

The energy-release law (10) is compared with experimental data by fitting the material parameters. For this purpose, we accomplished experiments with glass-polyester composite plates. Also, we compared the form (10) with the acoustic-emission measurements for other materials. Correlating the results for different samples, we notice that more disordered materials are characterized by larger values of the log-frequency ω and by smaller absolute values of the critical index $\Delta = \alpha - 1$. This index is negative, which reflects the divergence of the energy release rate at the moment of rupture. These observations confirm that the fracturing process is analogous to a critical phenomenon, and the point of rupture is somewhat equivalent to a critical point.

The knowledge of the general law of energy release for stressed composite materials not only clarifies the qualitative physical features of the fracturing process but can also be useful for a quantitative description of the behaviour of such materials and, hopefully, even for predicting the global failure of the latter. The possibility of predicting the time of global materials failure would be of great importance for industrial applications.

Appendix

Here we give a brief account of the basic ideas of the self-similar approximation theory, employed in this paper for deriving the general law of energy release. We shall stress only the principal points, while all technicalities can be found in the cited references.

Suppose we aim at finding a function $f(x)$ whose exact form is not known, but for which we can get a sequence $\{f_k(x)\}$ of approximate expressions numbered by $k = 0, 1, 2, \ldots$. For the time being, the physical nature of the function $f(x)$ is of no importance. The approach is very general and can be applied to arbitrary functions. The sequence $\{f_k(x)\}$ can be obtained by a kind of perturbation theory or by an iterative procedure. The standard situation in physical applications is that the sequence $\{f_k(x)\}$ either very poorely converges or, as in the majority of cases, fastly diverges. For example, the approximants $f_k(x)$ can be valid only for asymptotically small $x \to 0$, which happens for the expansion (6), but have no sense for the finite values of x. How could one extrapolate $f_k(x)$ to the region of their finite variables?

The first pivotal idea is the introduction of *control functions* (Yukalov, 1976). Let us reorganize the sequence $\{f_k(x)\}$ to another sequence $\{F_k(x, u)\}$ of the approximants

$$F_k(x, u) = \mathcal{R}\{f_k(x)\}$$

by means of a transformation $\mathcal{R}$ introducing a set of trial parameters u. The transformation $\mathcal{R}$ is assumed to have an inverse $\mathcal{R}^{-1}$, such that

$$\mathcal{R}^{-1}\{F_k(x, u)\} = f_k(x) \; .$$

Then, in each approximation order k, the trial parameters u are to be replaced by the control functions $u_k(x)$, yielding $F_k(x, u_k(x))$. The control functions are chosen so that to make convergent the sequence $\{\tilde{f}_k(x)\}$ of the *optimized approximants*

$$\tilde{f}_k(x) \equiv \mathcal{R}^{-1}\{F_k(x, u_k(x)\} \; . \qquad (A.1)$$

Note that, if necessary, some part of control functions can be defined not before the inverse transformation $\mathcal{R}^{-1}$ but after it, in order that the optimized approximants would possess the desired symmetry properties or satisfy boundary conditions. To this end, one may separate the trial parameters into two groups, say u and s. Then one defines a transformation $\mathcal{R}_s$ giving

$$F_k(x, u, s) = \mathcal{R}_s\{f_k(x)\} \; .$$

Now, generalizing the form (A.1), the optimized approximants are specified as

$$\tilde{f}_k(x) \lim_{s \to s_k(x)} \mathcal{R}^{-1}\{F_k(x, u_k(x), s)\} \; . \qquad (A.2)$$

The method of obtaining a convergent sequence $\{\tilde{f}_k(x)\}$ of the optimized approximants (A.1) or (A.2) is called the *optimized perturbation theory*. This approach is nowadays widely used in various applications (see survey in Yukalov and Yukalova, 2002).

In order to improve further the accuracy of calculations and to obtain an effective tool for controlling the convergence of the approximation sequences, it has been necessary to generalize the approach based on the usage of control functions. The new principal idea has been to interpret the passage from one successive approximation to another as the motion on the manifold of approximants, with the approximation order playing the role of discrete time. This motion can be formalized by means of *group self-similarity* (Yukalov, 1990, 1991, 1992). With this aim in view, we need to consider again the transformed sequence $\{F_k(x,u)\}$. Defining the function $x_k(\varphi)$ by the *reonomic constraint*

$$F_0(x,u_k(x)) = \varphi \; , \qquad x = x_k(\varphi) \; ,$$

we introduce

$$y_k(\varphi) \equiv F_k(x_k(\varphi), u_k(x_k(\varphi))) \; . \qquad (A.3)$$

Conversely, from $y_k(\varphi)$ we can return back to

$$F_k(x,u_k(x)) = y_k(F_0(x,u_k(x))) \; .$$

All functions $y_k(\varphi)$, with $k = 0, 1, 2, \ldots$, pertain to a linear normed space, which is complete in the sense of the norm convergence. We shall denote this Banach space by $\mathcal{B}$. The transformation $y_k : \mathcal{B} \rightarrow \mathcal{B}$ is an endomorphism of $\mathcal{B}$, with a unitary element given by $y_0(\varphi) = \varphi$. By construction, the sequences $\{y_k(\varphi)\}$ and $\{F_k(x,u_k(x))\}$ are bijective. The motion in $\mathcal{B}$ is presentable as the property of group self-similarity

$$y_{k+p}(\varphi) = y_k(y_p(\varphi)) \; , \qquad (A.4)$$

which is a necessary condition for the fastest convergence of the sequence $\{y_k(\varphi)\}$. The semigroup property (A.4) defines a dynamical system in discrete time, which is termed the approximation cascade. The cascade velocity is

$$v_k(\varphi) = y_k(\varphi) - y_{k-1}(\varphi) \; .$$

The self-similar approximation theory has its name being based on the group self-similarity (A.4). The fixed point $y^*(\varphi)$ of the approximation cascade $\{y_k | k = 0, 1, 2 \ldots\}$ corresponds to the sought function $f(x)$. An approximate fixed point $y_k^*(\varphi)$ is called a quasifixed point. Embedding the cascade $\{y_k\}$ into a flow and integrating the evolution equation, we get the *evolution integral*

$$\int_{y_{k-1}^*}^{y_k^*} \frac{d\varphi}{v_k(\varphi)} = \tau_k \; ,$$

in which τ_k is a control time required to rich the quasifixed point $y_k^*(\varphi)$. If the latter is found, then

$$F_k^*(x,u_k(x)) \equiv y_k^*(F_0(x,u_k(x)))$$

is also known. From here, we obtain the self-similar approximant

$$f_k^*(x) \equiv \mathcal{R}^{-1}\{F_k^*(x,u_k(x))\} \; . \qquad (A.5)$$

Similarly to the generalization (A.2), we may define a transformation $\mathcal{R}_s$, which introduces a set s of control functions to be chosen so that to satisfy some symmetry or boundary conditions. All procedure of constructing the approximation cascade, whose trajectory $\{y_k(\varphi, s)\}$ is bijective to the sequence $\{F_k(x, u_k(x), s)\}$, is the same as described above and results in the self-similar approximant:

$$f_k^*(x) \equiv \lim_{s \to s_k(x)} \mathcal{R}_s^{-1}\{F_k^*(x, u_k(x), s)\} \; . \qquad (A.6)$$

The self-similar approximants (A.5) and (A.6) are essentially more accurate than the optimized approximants (A.1) and (A.2).

It turned out that among different possible transformations $\mathcal{R}_s$ a very powerful is that one giving the *fractal transforms* (Yukalov and Gluzman, 1997-a-b, 1998-a; Yukalov, 2000). These are defined as

$$F_k(x, s) \equiv \mathcal{R}_s \; f_k(x) \equiv x^s f_k(x) \; .$$

The inverse transformation is

$$f_k(x) = \mathcal{R}_s^{-1} \; F_k(x, s) \equiv x^{-s} \; F_k(x, s) \; .$$

When applied to an asymptotic approximation

$$f_k(x) = 1 + \sum_{n=1}^{k} a_n x^{\alpha_n} \; , \qquad (A.7)$$

the fractal transformation gives

$$F_k(x, s) = x^s + \sum_{n=1}^{k} a_n \; x^{\alpha_n + s} \; .$$

The sequence $\{F_k(x, s)\}$ of the fractal transforms enjoys better convergence properties then the initial sequence $\{f_k(x)\}$. Expression (A.7) is the standard form usually arising in different variants of perturbation theory. This form is asymptotic with respect to $x \to 0$ and, as a rule, it has no sense for finite x, as in the case of expansion (7). The order of the powers α_n in the series (A.7) can, in general, be arbitrary, although usually they are arranged so that $\alpha_n < \alpha_{n+1}$, when α_n are real or $|\alpha_n| < |\alpha_{n+1}|$, when α_n are complex. Following the described steps of the self-similar approximation theory, with the usage of the fractal transforms, for the approximants (A.6), we get the recurrence relation

$$f_k^*(x) = \left\{ [f_{k-1}^*(x)]^{\delta_k} + A_k \; x^{\alpha_k} \right\}^{1/\delta_k} \; , \qquad (A.8)$$

in which $A_k \equiv a_k \delta_k \tau_k$ and $\delta_k \equiv -\alpha_k / s_k$. The relation (A.8) serves as a source for deriving the final form of the self-similar approximants. For deriving the self-similar exponential approximants, employed in the present paper, we notice that the series (A.7) can be reorganized to the iterative form

$$f_k(x) = 1 + z_1(1 + z_2(\ldots(1 + z_k))\ldots) \; , \qquad (A.9)$$

where

$$z_n = z_n(x) = \frac{a_n}{a_{n-1}}\; x^{\alpha_n - \alpha_{n-1}} \; .$$

Applying k times the described procedure to expression (A.9), with respect to variables z_n, we use the known limit

$$\lim_{s\to\pm\infty} \left(1 + \frac{\tau}{s}\; z \right)^s = \exp(\tau z) \; .$$

As a result, we obtain the self-similar exponential approximants

$$f_k^*(x) = \exp\left(c_1\; x^{\nu_1}\; \exp\left(c_2\; x^{\nu_2} \ldots \exp(c_k\; x^{\nu_k}) \right) \ldots \right) \; , \qquad (A.10)$$

in which the controllers are

$$c_n \equiv \frac{a_n}{a_{n-1}}\; \tau_n \qquad \nu_n \equiv \alpha_n - \alpha_{n-1} \; .$$

This way of deriving the expression (A.10) from an initially given asymptotic expansion (A.7) has been employed in obtaining the law (1) from the log-periodic approximation (7). The principal point is that the expansions (6) or (7) and (A.7) are only asymptotic, being valid solely for $x \to 0$; while the laws (10) and (A.10) being the self-similar extrapolations, are justified for finite values of the variable x.

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Figure captions

Figure 1. Synoptic diagram - Stressed composite material instrumented with piezoelectric transducers. The electric signals are recorded in real-time on computer.

Figure 2. Dimensionless energy release (10) fitted to our experimental data.

Figure 3. Dimensionless energy release (10), fitted to the experimental data of Anifrani *et al.*